Embeddings of group rings and L^2-invariants

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PhD defence

January 20, 2021

From rings. . .

What do these algebraic objects have in common?

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{Z}[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{Z}\}$$

$$\mathcal{O}(\mathbb{C}) = \{\text{holomorphic functions on } \mathbb{C}\}$$

From rings to fields

What do these algebraic objects have in common?
They are rings that admit fields of fractions.

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$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\}$$

$$\mathbb{Z}[x] = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in \mathbb{Z}\}$$
$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; \begin{array}{l} p(x) = a_n x^n + \cdots + a_1 x + a_0 \\ q(x) = b_m x^m + \cdots + b_1 x + b_0 \end{array}, a_i, b_j \in \mathbb{Z}, q \neq 0 \right\}$$

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Question: What about non-commutative rings?

Group rings

Definition

For a ring R and a group G, the group ring $R[G]$ is

$$\{\lambda_1 g_1 + \cdots + \lambda_n g_n \mid \lambda_i \in R, g_i \in G\}$$

with addition and multiplication extended R-linearly from G.

In the following, we consider $\mathbb{Z}[G]$ or $K[G]$ for a subfield K of $\mathbb{C}$.

- If G is finite, $K[G]$ is well understood (representation theory).
- If G is infinite, not much is known in general.

Malcev problem

If G is a torsion-free group, does KG embed into a division ring?

Betti numbers

- X: CW-complex of finite type

Definition

The n-th Betti number of X is

$$b_n(X) := \dim_{\mathbb{Q}} H_n(X) \in \mathbb{N}$$

Equivariant Betti numbers

- G: discrete, usually infinite group
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Most classical candidates either only depend on $G\backslash X$ or can be infinite:

$$\dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}[G]} H_n(X), \quad \dim_{\mathbb{Q}} H_n(\mathbb{Q} \otimes_{\mathbb{Z}[G]} C_*(X)),$$
$$b_n(G\backslash X), \quad b_n(G\backslash(X \times EG)), \quad \dots$$

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Need:
Well-behaved ring with a map from $\mathbb{Z}[G]$ and a dimension function

The L^2-machine

Ingredient #1:

$$\begin{array}{ccc} \mathbb{Z}[G] & \hookrightarrow & \mathcal{R}_{\mathbb{Z}[G]} \quad \ast\text{-regular} \\ \downarrow & & \downarrow \\ \mathcal{N}(G) & \hookrightarrow & \mathcal{U}(G) \end{array}$$

$\mathcal{R}_{K[G]}$ *can be constructed analogously.*

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- takes values in $[0, \infty]$
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Definition

The n-th L^2-Betti numbers of a G-CW-complex X of finite type is

$$b_n^{(2)}(X; G) := \dim_{\mathcal{R}_{\mathbb{Z}[G]}} H_n(\mathcal{R}_{\mathbb{Z}[G]} \otimes_{\mathbb{Z}[G]} C_*(X)) \in [0, \infty)$$

The L^2-machine for $G = \mathbb{Z}$

$$\begin{array}{ccccc}
\mathbb{Z}[z,z^{-1}] & & & & \mathbb{Q}(z) \\
& \cong & & \cong & \\
& \mathbb{Z}[\mathbb{Z}] & \hookrightarrow & \mathcal{R}_{\mathbb{Z}[\mathbb{Z}]} & \\
& \downarrow & & \downarrow & \\
& \mathcal{N}(\mathbb{Z}) & \hookrightarrow & \mathcal{U}(\mathbb{Z}) & \\
& \cong & & \cong & \\
L^\infty(S^1) & & & & L(S^1)
\end{array}$$

L^2-Betti numbers for $G = \mathbb{Z}$

$$b_n^{(2)}(X;\mathbb{Z}) = \dim_{\mathbb{Q}(z)} H_n(\mathbb{Q}(z) \otimes_{\mathbb{Z}[z,z^{-1}]} C_*(X)) \in \mathbb{Z}$$

The strong Atiyah conjecture

Strong Atiyah conjecture for G over $\mathbb{Q}$

Let G be a group with

$$\operatorname{lcm}(G) := \operatorname{lcm}\{|F| \mid F \leqslant G, |F| < \infty\} < \infty.$$

Then for every G-CW-complex X of finite type

$$b_n^{(2)}(X; G) \in \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

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The strong Atiyah conjecture is known for

- free-by-{elementary amenable group},
- residually {torsion-free elementary amenable} groups,
- fundamental groups of (most) 3-manifolds,
- one-relator groups,
- ...

Consequences of the strong Atiyah conjecture

Theorem

If G is torsion-free, then it satisfies the strong Atiyah conjecture over $\mathbb{Q}$ if and only if $\mathcal{R}_{\mathbb{Z}[G]}$ is a division ring.

Corollary

For torsion-free groups, the strong Atiyah conjecture implies a positive solution to the Malcev problem: $\mathbb{Z}G$ embeds into the division ring $\mathcal{R}_{\mathbb{Z}[G]}$.

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Questions:

- What if G has torsion?
- What can be said about $\mathcal{R}_{K[G]}$ for $K \subseteq \mathbb{C}$?

Groups with torsion I

Algebraic Atiyah conjecture for G over K (Jaikin-Zapirain)

The composition

$$\bigoplus_{F \leqslant G, |F| < \infty} K_0(K[F]) \to K_0(K[G]) \to K_0(\mathcal{R}_{K[G]})$$

is surjective.

Theorem (Knebusch, Linnell, Schick (plus $*$-regular rings))

The algebraic Atiyah conjecture for G over K holds if and only if $\mathcal{R}_{K[G]}$ is semisimple with an "Atiyah-expected" Artin–Wedderburn decomposition. In particular, the number of simple summands of $\mathcal{R}_{\mathbb{C}[G]}$ agrees with the number of finite conjugacy classes of finite order elements of G.

Groups with torsion II

Theorem (Jaikin-Zapirain)

If the strong Atiyah conjecture for a sofic group G holds over $\overline{\mathbb{Q}}$, then it holds over all $K \subseteq \mathbb{C}$.

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Theorem (M.)

If the algebraic Atiyah conjecture for a sofic group G holds over $\overline{\mathbb{Q}}$, then it holds over all $K \subseteq \mathbb{C}$ with $\mathrm{lcm}(G)$-th roots of unity.

Theorem (M.)

Let G be a sofic group and $K \subseteq \mathbb{C}$ a field of infinite transcendence degree over $\mathbb{Q}$. Then $\mathcal{R}_{K[G]}$ is unit-regular.

What makes $\mathcal{R}_{\mathbb{Z}[G]}$ special?

If all L^2-Betti numbers of a space vanish, one can define:

- universal L^2-torsion,
- twisted L^2-Euler characteristics,
- and the L^2-polytope.

Question: Does this require the analytic nature of $\mathcal{R}_{\mathbb{Z}[G]}$?

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If all L^2-Betti numbers of a space vanish, one can define:

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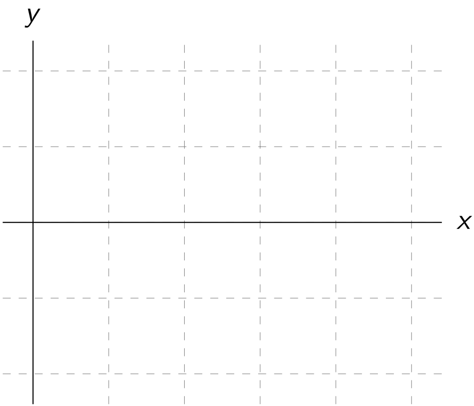
Theorem (Kielak, M.)

For any ring homomorphism $\mathbb{Z}[G] \to D$ to a division ring, analogues of these invariants can be defined that satisfy most of the known purely algebraic properties of L^2-invariants.*

The Friedl–Tillmann polytope

$$\pi = \langle x, y \mid R \rangle$$

$$R := yx^4yx^{-1}y^{-1}x^2y^{-1}x^{-2}y^2xy^{-1}xy^{-1}x^{-1}y^{-2}x^{-3}y^2x^{-1}yxy^{-1}x^{-1}$$



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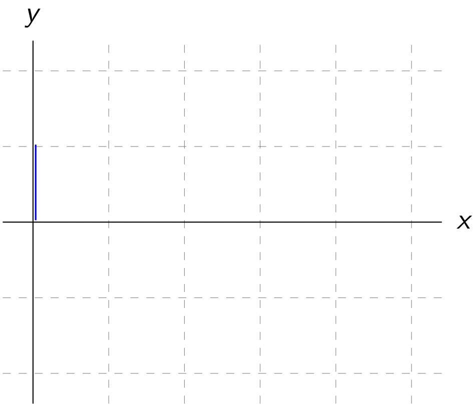

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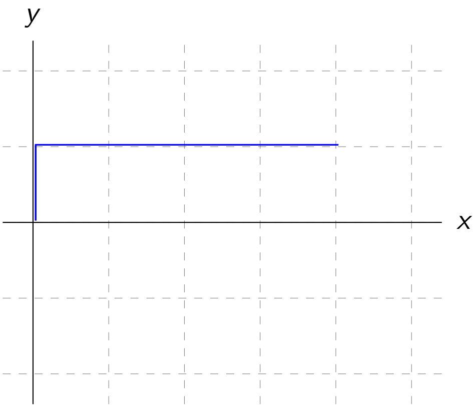

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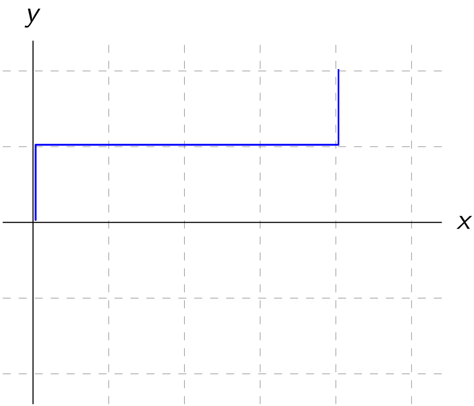

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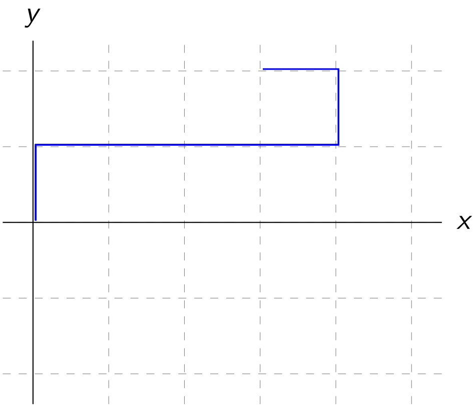

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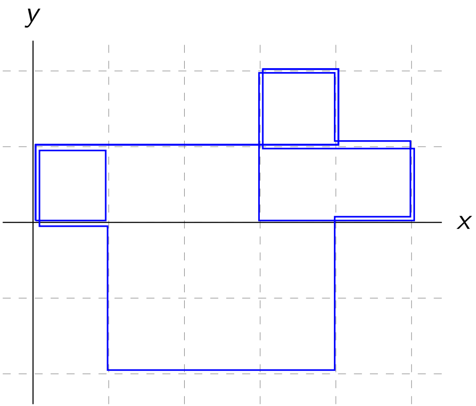

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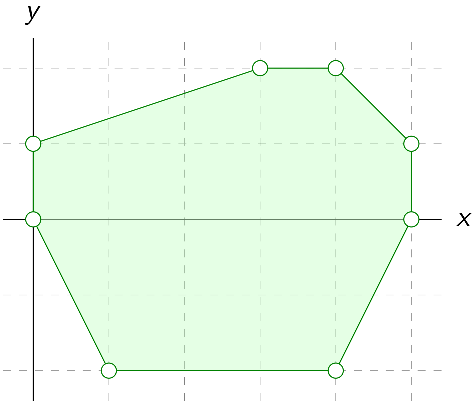

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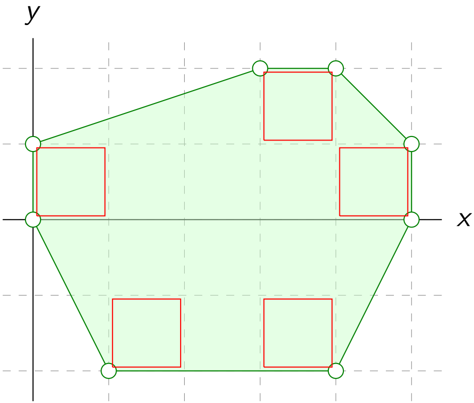

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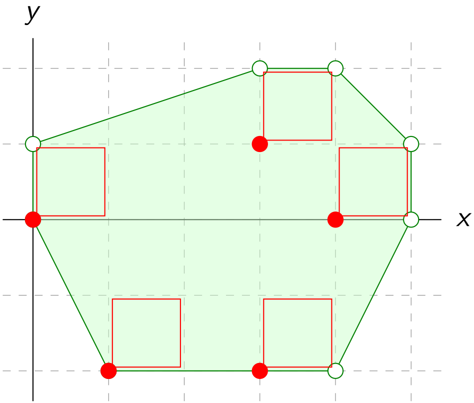

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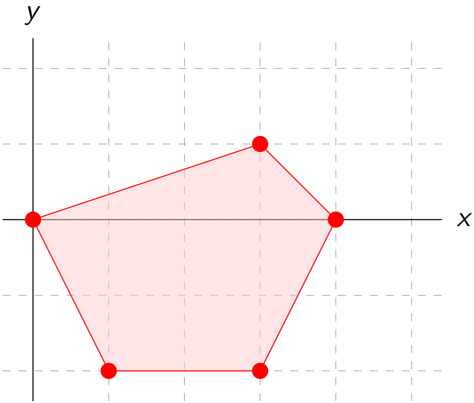

$\mathcal{P}_\pi$

The Friedl–Tillmann polytope as a group invariant

Conjecture (Friedl, Tillmann)

The polytope $\mathcal{P}_\pi$ is an invariant of the group (up to translation).

Theorem (Friedl, Tillmann)

✓ *if G_π is residually $\{$torsion-free elementary amenable$\}$.*

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Theorem (Friedl, Lück)

✓ *if G_π is torsion-free and satisfies the strong Atiyah conjecture.*

Meanwhile, the strong Atiyah conjecture has been proved for one-relator groups by López-Álvarez and Jaikin-Zapirain.

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Theorem (Kielak, M.)

✓ *for all two-generator one-relator groups.*

$\mathcal{R}_{K[G]}$ as a (pseudo-)Sylvester domain

Definition

An $n \times n$-matrix M is full if $M = PQ$ implies that P has at least n columns. It is stably full if $M \oplus \mathrm{Id}_r$ is full for all $r \geqslant 0$.

Invertible matrices and non-zero 1×1-matrices are stably full.

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Invertible matrices and non-zero 1×1-matrices are stably full.

Definition

A ring R is called a (pseudo-)Sylvester domain if it embeds into a division ring D over which all (stably) full R-matrices become invertible.

If this is the case, then D is (up to isomorphism over R) the division ring over which the most R-matrices become invertible, called the universal division ring of fractions of R.

$\mathcal{R}_{K[G]}$ as a (pseudo-)Sylvester domain

Theorem (López-Álvarez, M.)

Let $K \subset \mathbb{C}$ be a field and G a free-by-$\{$infinite cyclic$\}$ group G. Then

- *stably full $K[G]$-matrices are invertible over $\mathcal{R}_{K[G]}$;*
- *full $K[G]$-matrices are invertible over $\mathcal{R}_{K[G]}$ if and only if every stably free $K[G]$-module is free.*

Examples

$\mathbb{Q}[\mathbb{Z}^2]$ ✓ $\mathbb{Q}[F_2 \times \mathbb{Z}]$ ✓ $\mathbb{Q}[\mathbb{Z} \rtimes \mathbb{Z}]$ ✗ $\mathbb{Q}[\langle x, y \mid x^3 = y^2 \rangle]$ ✗

Publications

- with Dawid Kielak, *Agrarian and L^2-invariants*, (2019), arXiv: 1809.08470 [math.AT].
- with Dawid Kielak, *The agrarian polytope of two-generator one-relator groups*, J. Lond. Math. Soc. (2) **102** (2020), 722–748, DOI: 10.1112/jlms.12334.
- with Diego López-Álvarez, *Pseudo-Sylvester domains and skew Laurent polynomials over firs*, (2020), arXiv: 2006.08454 [math.RA].